What is a **Probability**?

**Probability** is a branch of mathematics that deals with calculating the likelihood of a given event to happen or not, which is expressed as a number from 0 to 1.

What is an **Event**?

An **Event** is any collection of outcomes of a procedure.

What is a **Simple Event**?

An **Event** that cannot be further broken down into simpler components.
What is a **Sample Space**?

**Sample Space** is a collection of all possible simple events of a procedure.

**Example:**
Find the sample space for the following procedures.

1. Single birth
2. Flip a coin twice
3. Flip a coin followed by rolling a four-sided die
Solution:

1. Single birth $\implies$ let’s use $B$ to denote a boy and $G$ to denote a girl, then the sample space is $\{B, G\}$.

2. Flip a coin twice $\implies$ let’s use $H$ to denote heads outcome and $T$ to denote tails outcome, then the sample space is $\{HH, HT, TH, TT\}$.

3. Flip a coin followed by rolling a four-sided die $\implies$ let’s use $H$ to denote heads outcome and $T$ to denote tails outcome along with numbers 1, 2, 3, 4 for the outcomes of the four-sided die then the sample space is $\{H1, H2, H3, H4, T1, T2, T3, T4\}$.
How do we find the **Probability** of an **Event**?

\[
\text{Probability}(\text{Desired Event}) = \frac{\text{The number of desired outcomes}}{\text{The number of all possible outcomes}}
\]

**Example:**

Consider a full-deck of playing cards shown below.

What is the probability of randomly drawing an ace?
What is the probability of randomly drawing a face card?
Solution:

Probability(Draw an ace) = \frac{\text{Number of aces}}{\text{Total number of cards}}
= \frac{4}{52} = \frac{1}{13}
\approx 0.077

Probability(Draw a face card) = \frac{\text{Number of face cards}}{\text{Total number of cards}}
= \frac{12}{52} = \frac{3}{13}
\approx 0.231
What are the properties of Probability?

Let $E$ be all possible events, $A$ be the desired event with $P(E)$ and $P(A)$ be the corresponding probabilities,

- $0 \leq P(A) \leq 1$
- $\sum P(E) = 1$
- $\bar{A}$ is the complement of the event $A$, which means not $A$.  
  
  - $P(\bar{A}) + P(A) = 1$, or $P(\bar{A}) = 1 - P(A)$
Example:

Which of the following values cannot be probabilities?

\[ \frac{7}{5}, -0.75, 125\% \]

Solution:

None of these values can be used to express the probabilities since they do not satisfy \( 0 \leq P(A) \leq 1 \).

Example:

Find \( P(\overline{A}) \) if \( P(A) = 0.05 \).

Solution:

Since \( P(\overline{A}) = 1 - P(A) \), so \( P(\overline{A}) = 1 - 0.05 \) then \( P(\overline{A}) = 0.95 \).
What is a **Sure Event**?

Event $A$ is considered a **Sure Event** if $P(A) = 1$.

**Example:**

Suppose you roll a normal die. What is the probability that you will get a number less than 7?

**Solution:**

The probability that you will get a number less than 7 is 1 since any outcome is a number less than 7. The event is a sure event.
What is an **Impossible Event**?

Event $A$ is considered an **Impossible Event** if $P(A) = 0$.

*Example:*

What is the probability that someone is born on February 30th?

*Solution:*

The probability that someone is born on February 30th is 0 since there is no such date on the calendar. The event is impossible.
What is a **Rare Event**?

Event $A$ is considered a **Rare Event** if $0 < P(A) \leq 0.05$.

**Example:**

What is the probability that randomly selected person has a birthday today?

**Solution:**

The probability that anyone randomly selected has a birthday today is $\frac{1}{365} \approx 0.003$ since that is less than $0.05$, it is a rare event.
Basic Probability Scale

- Impossible
- Very unlikely
- Unlikely
- Even chance
- Likely
- Very likely
- Certain

Probability Scale:
- 0: Impossible
- 1/4
- 1/2: Even chance
- 3/4
- 1: Certain
Example:

Suppose a red fair die and a white fair die is rolled. The display below shows all possible outcomes.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,1)</td>
<td>(2,1)</td>
<td>(3,1)</td>
<td>(4,1)</td>
<td>(5,1)</td>
<td>(6,1)</td>
</tr>
<tr>
<td>2</td>
<td>(1,2)</td>
<td>(2,2)</td>
<td>(3,2)</td>
<td>(4,2)</td>
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<td>(6,2)</td>
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<tr>
<td>3</td>
<td>(1,3)</td>
<td>(2,3)</td>
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<td>(6,3)</td>
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<tr>
<td>6</td>
<td>(1,6)</td>
<td>(2,6)</td>
<td>(3,6)</td>
<td>(4,6)</td>
<td>(5,6)</td>
<td>(6,6)</td>
</tr>
</tbody>
</table>

1. List all possible sums.
2. What is the probability that the sum of the outcomes is 1?
3. What is the probability that the sum of the outcomes is between 2 and 12, inclusive?
Solution:
1. List all possible sums ⇒ \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}
2. \(P(\text{Sum} = 1) = 0\) since there is no outcome with the sum of 1.
3. \(P(2 \leq \text{Sum} \leq 12) = 1\) since the sum of any outcomes is between 2 and 12, inclusive.

Example:
Use the last example to complete the following table

<table>
<thead>
<tr>
<th>Sum</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(Sum)</td>
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</tr>
</tbody>
</table>

then verify that \(\sum P(\text{Sum}) = 1\).
Solution:

There are 36 outcomes altogether,

\[ P(\text{Sum} = 2) = P((1, 1)) = \frac{1}{36}, \quad P(\text{Sum} = 12) = P((6, 6)) = \frac{1}{36}, \]

\[ P(\text{Sum} = 3) = P((1, 2), (2, 1)) = \frac{2}{36}, \quad P(\text{Sum} = 11) = P((6, 5), (5, 6)) = \frac{2}{36}, \]

\[ P(\text{Sum} = 4) = P((1, 3), (2, 2), (3, 1)) = \frac{3}{36}, \]

We continue this to get the rest of the probabilities.

<table>
<thead>
<tr>
<th>Sum</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(Sum)</td>
<td>(\frac{1}{36})</td>
<td>(\frac{2}{36})</td>
<td>(\frac{3}{36})</td>
<td>(\frac{4}{36})</td>
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</tbody>
</table>

It is easier to verify that \(\sum P(\text{Sum}) = 1\) if these probabilities are not reduce.
There's a 100% chance of me teaching you probability.