Chapter 6
Normal Probability Distributions

6-2 The Standard Normal Distribution
6-3 Applications of Normal Distributions
6-4 Sampling Distributions and Estimators
6-5 The Central Limit Theorem
6-6 Normal as Approximation to Binomial

Overview

- Continuous random variable
- Normal distribution

Definitions

- Uniform Distribution is a probability distribution in which the continuous random variable values are spread evenly over the range of possibilities; the graph results in a rectangular shape.

Because the total area under the density curve is equal to 1, there is a correspondence between area and probability.

Using Area to Find Probability

Is the total area equal to 1?
Heights of Adult Men and Women

Why is the red curve on the right?
Why does the blue curve have higher height?

Definition
Standard Normal Distribution:
a normal probability distribution that has a
mean of 0 and a standard deviation of 1, and the
total area under its density curve is equal to 1.

Notation
\( P(a < z < b) \)
denotes the probability that the \( z \) score is
between \( a \) and \( b \)
\( P(z > a) \)
denotes the probability that the \( z \) score is
greater than \( a \)
\( P(z < a) \)
denotes the probability that the \( z \) score is
less than \( a \)

Table A-2
- Inside front cover of textbook
- Formulas and Tables card
- Appendix
To find:

**z Score**
the distance along horizontal scale of the standard normal distribution; refer to the leftmost column and top row of Table A-2.

**Area**
the region under the curve; refer to the values in the body of Table A-2.

---

**Example:** If thermometers have an average (mean) reading of 0 degrees and a standard deviation of 1 degree for freezing water, and if one thermometer is randomly selected, find the probability that, at the freezing point of water, the reading is less than 1.58 degrees.

\[ P(z < 1.58) = 0.9429 \]

The probability that the chosen thermometer will measure freezing water less than 1.58 degrees is 0.9429.

---

**Using TI:**

**Standard Normal Distribution**

\[ P(z < a) \]

1) 2nd VARS (DISTR)
2) Arrow down to normalcdf(
3) enter
4) VSNN, a, 0, 1) enter

Mean VSNN: Very Small Negative Number
Using TI: Standard Normal Distribution

**Example:** Find \( P(z < 1.58) \)

1) Select 2nd, VARS, arrow down to get to \texttt{normalcdf(} enter to select

2) key in \texttt{normalcdf(-1000, 1.58,0,1)}

This result was obtained earlier by directly using the Standard Normal Distribution table.

---

**Example:** If thermometers have an average (mean) reading of 0 degrees and a standard deviation of 1 degree for freezing water, and if one thermometer is randomly selected, find the probability that it reads (at the freezing point of water) above –1.23 degrees.

\( P(z > -1.23) = 0.8907 \)

The probability that the chosen thermometer with a reading above –1.23 degrees is 0.8907.

---

**Using TI: Standard Normal Distribution**

\( P(z > a) \)

1) 2nd VARS( \texttt{DISTR} )

2) Arrow down to \texttt{normalcdf(} enter to select

3) enter

4) \( a, \text{ VLPN}, 0, 1 \) enter

\texttt{VLPN} \rightarrow \text{Very Large Positive Number}
Using TI:
Standard Normal Distribution

Example: Find \( P( z > -1.23 ) \)
3) Enter to execute this operation and get the final answer.

This result was obtained earlier by directly using the Standard Normal Distribution table.

Example: A thermometer is randomly selected. Find the probability that it reads (at the freezing point of water) between -2.00 and 1.50 degrees.

\[
\begin{align*}
P( z < -2.00 ) &= 0.0228 \\
P( z < 1.50 ) &= 0.9332 \\
P( -2.00 < z < 1.50 ) &= 0.9332 - 0.0228 = 0.9104
\end{align*}
\]

The probability that the chosen thermometer has a reading between -2.00 and 1.50 degrees is 0.9104.

Using TI:
Standard Normal Distribution

\( P(a < z < b) \)
1) 2nd VARS ( DISTR )
2) Arrow down to normalcdf(
3) enter
4) a , b , 0 , 1 ) enter

Mean

Standard Deviation

If many thermometers are selected and tested at the freezing point of water, then 91.04% of them will read between -2.00 and 1.50 degrees.

Using TI:
Standard Normal Distribution

Example: Find \( P( -2.00 < z < 1.50 ) \)
1) Select 2nd, VARS, arrow down to get to normalcdf( enter to select
2) key in \(-2.00, 1.50, 0, 1\) enter

This result was obtained earlier by directly using the Standard Normal Distribution table.
Finding a $z$-score when given a probability Using Table A-2

1. Draw a bell-shaped curve, draw the centerline, and identify the region under the curve that corresponds to the given probability. If that region is not a cumulative region from the left, work instead with a known region that is a cumulative region from the left.

2. Using the cumulative area from the left, locate the closest probability in the body of Table A-2 and identify the corresponding $z$-score.

Using TI: Standard Normal Distribution

Example: Find the $z$ - score if $P( z < a )=0.95$

1) Select 2nd, VARS, arrow down to get to invNorm( enter to select

2) key in 0.95 , 0 , 1 )

3) Enter to execute this operation and get the final answer.

This result was obtained earlier by directly using the Standard Normal Distribution table.
Finding $z$ Scores when Given Probabilities

Figure 5-11
Finding the Bottom 2.5% and Upper 2.5%

(One $z$ score will be negative and the other positive)

Finding $z$ Scores when Given Probabilities

Using TI: Standard Normal Distribution

Example: Find the $z$-score if $P( z < a )=0.025$

1) Select 2nd, VARS, arrow down to get to `invNorm`
2) Key in $0.025, 0, 1$
3) Enter to execute this operation and get the final answer.

This result was obtained earlier by directly using the Standard Normal Distribution table.

Nonstandard Normal Distributions

If $\mu \neq 0$ or $\sigma \neq 1$ (or both), we will convert values to standard scores using Formula 5-2, then procedures for working with all normal distributions are the same as those for the standard normal distribution.

Formula 5-2  
$$z = \frac{x - \mu}{\sigma}$$
Converting to Standard Normal Distribution

\[ z = \frac{x - \mu}{\sigma} \]

**Figure 5-12**

Probability of Sitting Heights Less Than 38.8 Inches

- The sitting height (from seat to top of head) of drivers must be considered in the design of a new car model. Men have sitting heights that are normally distributed with a mean of 36.0 in. and a standard deviation of 1.4 in. (based on anthropometric survey data from Gordon, Clauser, et al.). Engineers have provided plans that can accommodate men with sitting heights up to 38.8 in., but taller men cannot fit. If a man is randomly selected, find the probability that he has a sitting height less than 38.8 in. Based on that result, is the current engineering design feasible?

**Probability of Sitting Heights Less Than 38.8 Inches**

\[ z = \frac{38.8 - 36.0}{1.4} = 2.00 \]

\[ P(z < 2) = 0.9772 \]

**Figure 5-13**

Using TI:
Non-Standard Normal Distribution

\[ P(x < a) \]

1) 2nd VARS( DISTR )
2) Arrow down to normalcdf( 
3) enter
4) VSNN, a, \( \mu \), \( \sigma \) enter

Mean Standard Deviation

VSNN \( \rightarrow \) Very Small Negative Number

Using TI:
Standard Normal Distribution

Example: Find \( P(x < 38.8) \) when \( \mu = 36.0 \) and \( \sigma = 1.4 \).

1) Select 2nd, VARS, arrow down to normalcdf( 
2) key in \( \text{normalcdf}(-1000, 38.8, 36.0, 1.4) \) enter to select

Mean Standard Deviation
Using TI:
Non-Standard Normal Distribution

Example: Find \( P(x < 38.8) \) when \( \mu = 36.0 \) and \( \sigma = 1.4 \).

3) Enter to execute this operation and get the final answer.

This result was obtained earlier by converting to the Standard Normal Distribution and then using the table.

Probability of Weight between 140 pounds and 211 pounds

In the Chapter Problem, we noted that the Air Force had been using the ACES-II ejection seats designed for men weighing between 140 lb and 211 lb. Given that women’s weights are normally distributed with a mean of 143 lb and a standard deviation of 29 lb (based on data from the National Health survey), what percentage of women have weights that are within those limits?
There is a 0.5302 probability of randomly selecting a woman with a weight between 140 and 211 lbs.

OR - 53.02% of women have weights between 140 lb and 211 lb.

Using TI:
Non-Standard Normal Distribution

\[ P(a < x < b) \]

1) 2nd, VARS, arrow down to normalcdf
2) Arrow down to normalcdf
3) enter
4) a, b, μ, σ) enter

Mean Standard Deviation

Finding a \( z \)-score when given a probability Using Table A-2

1. Draw a bell-shaped curve, draw the centerline, and identify the region under the curve that corresponds to the given probability. If that region is not a cumulative region from the left, work instead with a known region that is a cumulative region from the left.

2. Using the cumulative area from the left, locate the closest probability in the body of Table A-2 and identify the corresponding \( z \)-score.
Cautions to keep in mind

1. Don't confuse \(z\) scores and areas. \(z\) scores are distances along the horizontal scale, but areas are regions under the normal curve. Table A-2 lists \(z\) scores in the left column and across the top row, but areas are found in the body of the table.

2. Choose the correct (right/left) side of the graph.

3. A \(z\) score must be negative whenever it is located to the left half of the normal distribution.

4. Areas (or probabilities) are positive or zero values, but they are never negative.

Procedure for Finding Values Using Table A-2 and Formula 5-2

1. Sketch a normal distribution curve, enter the given probability or percentage in the appropriate region of the graph, and identify the \(x\) value(s) being sought.

2. Use Table A-2 to find the \(z\) score corresponding to the cumulative left area bounded by \(x\). Refer to the BODY of Table A-2 to find the closest area, then identify the corresponding \(z\) score.

3. Using Formula 5-2, enter the values for \(\mu\), \(\sigma\), and the \(z\) score found in step 2, then solve for \(x\).

\[ x = \mu + (z \cdot \sigma) \] (Another form of Formula 5-2)

(If \(z\) is located to the left of the mean, be sure that it is a negative number.)

4. Refer to the sketch of the curve to verify that the solution makes sense in the context of the graph and the context of the problem.

---

Find \(P_{98}\) for Hip Breadths of Men

\(z = 2.05\)

Figure 5-15

\[ x = \mu + (z \cdot \sigma) \]
\[ x = 14.4 + (2.05 \cdot 1.0) \]
\[ x = 16.45 \]

Seats designed for a hip breadth up to 16.5 in. will fit 98% of men.
Using TI: Non-Standard Normal Distribution

\[ P(x < a) = LMA \]

1) 2nd VARS (DISTR)
2) Arrow down to invNorm
3) Enter
4) LMA, \( \mu \), \( \sigma \) enter

Mean \( \rightarrow \) Left Most Area

Example: Find the x-score if \( P(x < a) = 0.98 \) when \( \mu = 14.4 \) and \( \sigma = 1.0 \).

1) Select 2nd, VARS, arrow down to get invNorm
2) Enter to select
3) Enter 0.98, 14.4, 1.0, Enter

Mean \( \rightarrow \) Standard Deviation

This result was obtained earlier by directly using the Standard Normal Distribution table.

Finding \( P_{0.05} \) for Grips of Women

\[ x = 27.0 + (-1.645 \times 1.3) = 24.8615 \]
Finding $P_{0.05}$ for Grips of Women

The forward grip of 24.9 in. (rounded) separates the top 95% from the others.

![Graph showing the separation of the top 95% from others.]

Figure 5-16

Using TI: Non-Standard Normal Distribution

Example: Find the $x$ - score if $P( x < a )=0.05$ when $\mu=27.0$ and $\sigma=1.3$.

1) Select 2nd, VARS, arrow down to get to `invNorm(` enter to select

2) key in $0.05, 27.0 , 1.3$)

This result was obtained earlier by directly using the Standard Normal Distribution table.

REMEMBER!

Make the $z$ score negative if the value is located to the left (below) the mean. Otherwise, the $z$ score will be positive.

Definition

Sampling Distribution of the mean is the probability distribution of sample means, with all samples having the same sample size $n$.

Definition

Sampling Variability:
The value of a statistic, such as the sample mean $\bar{x}$, depends on the particular values included in the sample.
Consider the population of 2, 4, and 6. Select sample of size 2 with replacement.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Sample Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, 2</td>
<td>2</td>
</tr>
<tr>
<td>2, 4</td>
<td>3</td>
</tr>
<tr>
<td>2, 6</td>
<td>4</td>
</tr>
<tr>
<td>4, 2</td>
<td>3</td>
</tr>
<tr>
<td>4, 4</td>
<td>4</td>
</tr>
<tr>
<td>4, 6</td>
<td>5</td>
</tr>
<tr>
<td>6, 2</td>
<td>4</td>
</tr>
<tr>
<td>6, 4</td>
<td>5</td>
</tr>
<tr>
<td>6, 6</td>
<td>6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sample Mean</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1/9</td>
</tr>
<tr>
<td>3</td>
<td>2/9</td>
</tr>
<tr>
<td>4</td>
<td>3/9</td>
</tr>
<tr>
<td>5</td>
<td>2/9</td>
</tr>
<tr>
<td>6</td>
<td>1/9</td>
</tr>
</tbody>
</table>

Consider the population of 2, 4, and 6. Select sample of size 2 with replacement.

Now use your calculator to compute the mean and standard deviation of the sample means.

- a) Enter sample means into L1
- b) Enter corresponding probabilities into L2.
- c) Stat, Calc, 1-var stat, L1, L2, enter

Now
- a) Enter element of the population into L3.
- b) Stat, Calc, 1-var stat, L1, L2, enter

Interpretation of Sampling Distributions

We can see that when using a sample statistic to estimate a population parameter, some statistics are good in the sense that they target the population parameter and are therefore likely to yield good results. Such statistics are called unbiased estimators.

Statistics that target population parameters: mean, variance, proportion

Statistics that do not target population parameters: median, range, standard deviation

Central Limit Theorem

Given:
1. The random variable $x$ has a distribution (which may or may not be normal) with mean $\mu$ and standard deviation $\sigma$.
2. Samples all of the same size $n$ are randomly selected from the population of $x$ values.

Conclusions:
1. The distribution of the sample means will, as the sample size increases, approach a normal distribution.
2. The mean of the sample means will be the population mean $\mu$.
3. The standard deviation of the sample means will approach $\frac{\sigma}{\sqrt{n}}$. 
Practical Rules

Commonly Used:

1. For samples of size \( n \) larger than 30, the distribution of the sample means can be approximated reasonably well by a normal distribution. The approximation gets better as the sample size \( n \) becomes larger.

2. If the original population is itself normally distributed, then the sample means will be normally distributed for any sample size \( n \) (not just the values of \( n \) larger than 30).

Notation

- the mean of the sample means \( \mu_x = \mu \)
- the standard deviation of sample mean \( \sigma_x = \frac{\sigma}{\sqrt{n}} \)
  (often called standard error of the mean)

Using TI

When using `normalcdf` with sample size \( n = 1 \), enter the following four entries in the order: `normalcdf( L V , R V , \mu , \sigma )`.

Example: \( P(x > 24) \) when \( \mu = 26 \) & \( \sigma = 1.5 \)

\( P(x > 24) = \text{normalcdf}(24,1000, 26,1.5) \)

Answer:

Using TI

When using `normalcdf` with sample size \( n > 1 \), enter the following four entries in the order:

`normalcdf( L V , R V , \mu , \frac{\sigma}{\sqrt{n}} )`.

Example:

\( P(\bar{x} > 24) \) when \( n = 10, \mu = 26 \) & \( \sigma = 1.5 \)

\( P(\bar{x} > 24) = \text{normalcdf}(24,1000, 26, \frac{1.5}{\sqrt{10}}) \)

Answer:

Example:

Given the population of men has normally distributed weights with a mean of 172 lb and a standard deviation of 29 lb,

a) if one man is randomly selected, find the probability that his weight is greater than 167 lb.

b) if 12 different men are randomly selected, find the probability that their mean weight is greater than 167 lb.

Example: Given the population of men has normally distributed weights with a mean of 172 lb and a standard deviation of 29 lb,

a) if one man is randomly selected, the probability that his weight is greater than 167 lb. is 0.5675.

Use your Calculator to verify this number.
Example: Given the population of men has normally distributed weights with a mean of 172 lb and a standard deviation of 29 lb,
b) if 12 different men are randomly selected, find the probability that their mean weight is greater than 167 lb.

\[ P(\bar{x} > 167) = 0.7257 \]

It is much easier for an individual to deviate from the mean than it is for a group of 12 to deviate from the mean.

Sampling Without Replacement

If \( n > 0.05N \)

\[ \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}} \]

finite population correction factor

Approximate a Binomial Distribution with a Normal Distribution if:

\[ np \geq 5 \]
\[ nq \geq 5 \]

then \( \mu = np \) and \( \sigma = \sqrt{npq} \)

and the random variable has a \( (normal) \) distribution.

Procedure for Using a Normal Distribution to Approximate a Binomial Distribution

1. Establish that the normal distribution is a suitable approximation to the binomial distribution by verifying \( np \geq 5 \) and \( nq \geq 5 \).
2. Find the values of the parameters \( \mu \) and \( \sigma \) by calculating \( \mu = np \) and \( \sigma = \sqrt{npq} \).
3. Identify the discrete value of \( x \) (the number of successes). Change the discrete value \( x \) by replacing it with the interval from \( x - 0.5 \) to \( x + 0.5 \). Draw a normal curve and enter the values of \( \mu \), \( \sigma \), and either \( x - 0.5 \) or \( x + 0.5 \), as appropriate.

continued

continued

4. Change \( x \) by replacing it with \( x - 0.5 \) or \( x + 0.5 \), as appropriate.
5. Find the area corresponding to the desired probability.
Definition

When we use the normal distribution (which is continuous) as an approximation to the binomial distribution (which is discrete), a **continuity correction** is made to a discrete whole number \( x \) in the binomial distribution by representing the single value \( x \) by the interval from \( x - 0.5 \) to \( x + 0.5 \).

Procedure for Continuity Corrections

1. When using the normal distribution as an approximation to the binomial distribution, always use the continuity correction.
2. In using the continuity correction, first identify the discrete whole number \( x \) that is relevant to the binomial probability problem.
3. Draw a normal distribution centered about \( \mu \), then draw a vertical strip area centered over \( x \). Mark the left side of the strip with \( x - 0.5 \), and mark the right side with \( x + 0.5 \). For \( x = 120 \), draw a strip from 119.5 to 120.5. Consider the area of the strip to represent the probability of discrete number \( x \).

4. Now determine whether the value of \( x \) itself should be included in the probability you want. Next, determine whether you want the probability of at least \( x \), at most \( x \), more than \( x \), fewer than \( x \), or exactly \( x \). Shade the area to the right or left of the strip, as appropriate; also shade the interior of the strip itself if and only if \( x \) itself is to be included. The total shaded region corresponds to probability being sought.

Finding the Probability of

“…………..”

120 Men Among 200 Accepted Applicants

\( x = \text{exactly} \ 120 \)

\( x = \text{at least} \ 120 \)
\( = 120, 121, 122, \ldots \)

\( x = \text{more than} \ 120 \)
\( = 121, 122, 123, \ldots \)

\( x = \text{at most} \ 120 \)
\( = 0, 1, \ldots 118, 119, 120 \)

\( x = \text{fewer than} \ 120 \)
\( = 0, 1, \ldots 118, 119 \)

Figure 5-24

Figure 5-25